CENTRE FOR QUANTUM TECHNOLOGY, SINGAPORE

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Quantum-style tensor network factorisation in stochastic networks

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What I want to get across

What a tensor network is (and why it is useful)

Tensor networks can be used for stochastic processes, as well as quantum etc.

Tensor networks could be essential for high variance

Tensor networks compress (1)

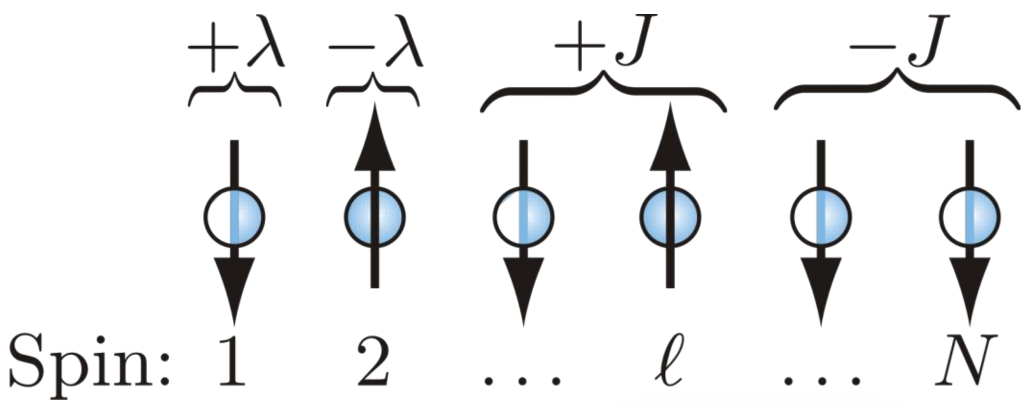
Consider the (un-normalized probability) distribution

$$P(\mathbf{z}) = \exp[-\beta E(\mathbf{z})] \quad \begin{cases} \mathbf{z} = (z_1, \ldots, z_N) \\ z_\ell \in \{-1, 1\} \end{cases}$$

where

$$E(\mathbf{z}) = -J \sum_{\ell=1}^{N} z_\ell z_{\ell+1} - \lambda \sum_{\ell=1}^{N} z_\ell$$

i.e. the Ising model



Tensor networks compress (2)

It factorises, one local object for each degree of freedom

$$E(\mathbf{z}) = \sum_{\ell=1}^{N} E^{[\ell] z_\ell z_{\ell+1}}$$

so

$$P(\mathbf{z}) = A^{[1] z_1} \dots A^{[\ell] z_\ell} \dots A^{[N] z_N}$$

$1 \qquad 1 \times \chi \qquad \chi \times \chi \qquad \chi \times 1$

$$\left(A^{[1] z_1}\right)_{\mu_1} = \exp[-\beta E^{[1] z_1 \mu_1}]$$

$$\left(A^{[\ell] z_\ell}\right)_{\mu_{\ell-1} \mu_\ell} = \delta_{z_\ell, \mu_{\ell-1}} \exp[-\beta E^{[\ell] z_\ell \mu_\ell}]$$

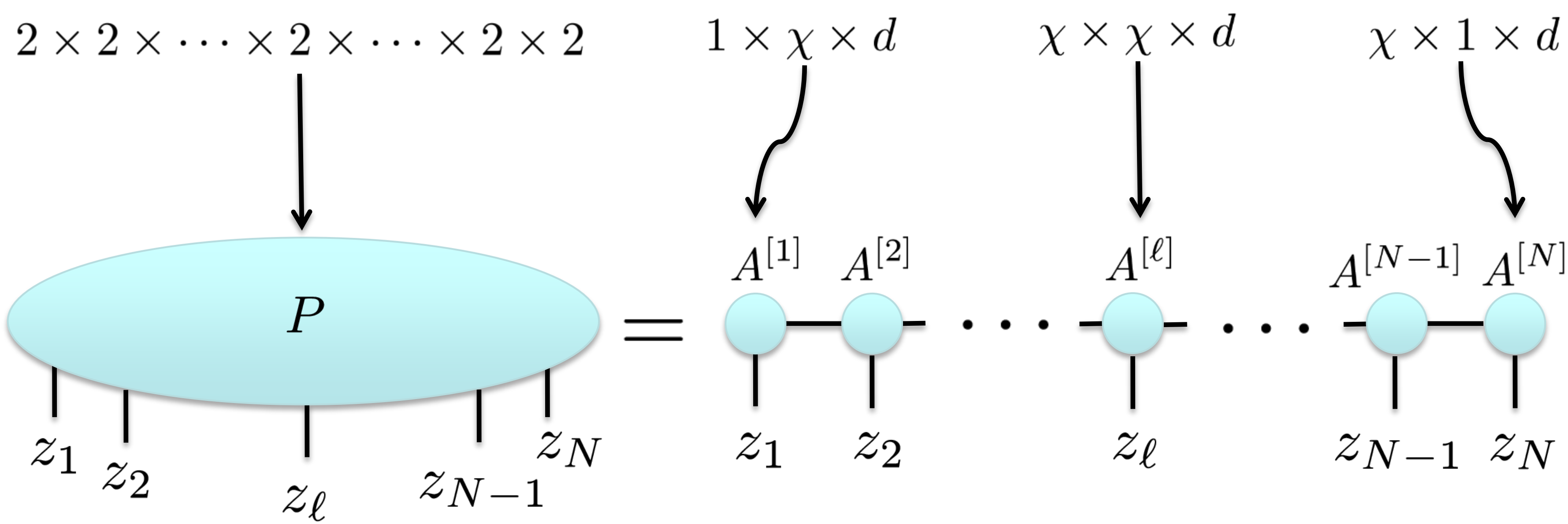
$$\left(A^{[N] z_N}\right)_{\mu_{N-1}} = \delta_{z_N, \mu_{N-1}}$$

$$\chi = 2$$

$$\mu_\ell \in \{-1, 1\}$$

Tensor networks compress (3)

This is a tensor factorisation



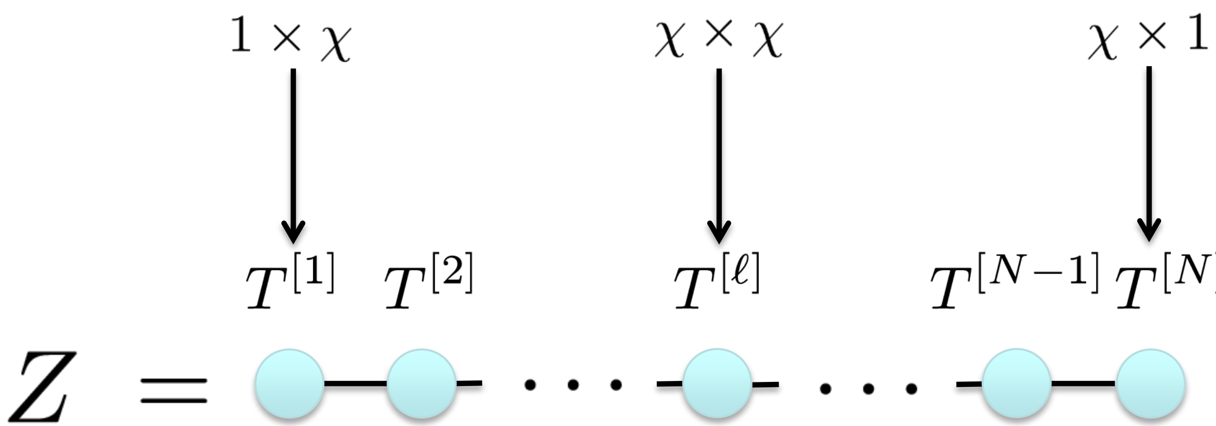
$$d^N \qquad N d \chi^2$$

$$\chi = 2$$
$$d = 2$$

You’ve probably seen this before

Transfer matrix method

$$Z = \sum_{\mathbf{z}} P(\mathbf{z}) = \left(\sum_{z_1} A^{[1]z_1}\right) \cdots \left(\sum_{z_N} A^{[N]z_1 N}\right) = T^{[1]} \dots T^{[N]}$$

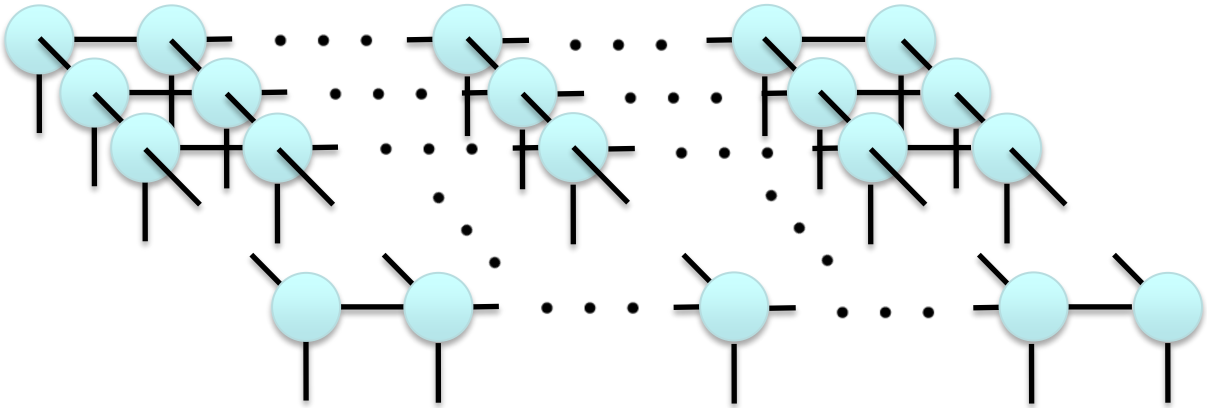


2D and beyond

Ising spins connected in a 2D arrangement:

$$E(\mathbf{z}) = -J \sum_{\langle \ell \ell' \rangle} z_\ell z_{\ell'} - \lambda \sum_\ell z_\ell$$

$P =$



$$N d \chi^4$$

What I'm not going to talk about: Evolution

Recall: Evolution operator

$$|P(t)\rangle = \mathrm{e}^{Ht}|P(0)\rangle$$

Assume: Nearest-neighbour

$$H = \sum^{N-1} h \ldots$$

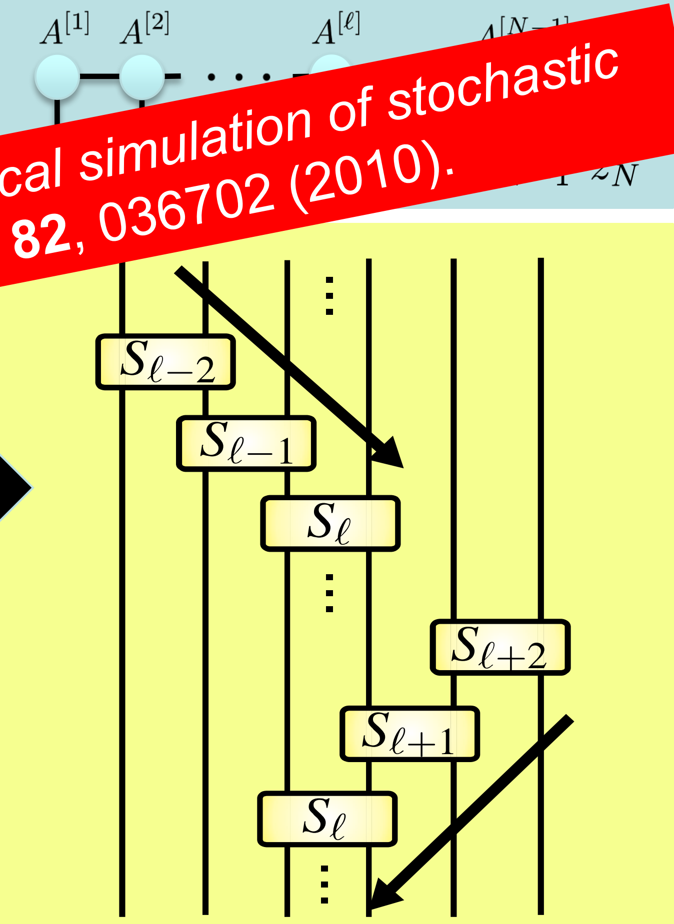
T. H. Johnson, S. R. Clark, and D. Jaksch, Dynamical simulation of stochastic systems using matrix product states, Phys. Rev. E **82**, 036702 (2010).

$$\ldots \mathrm{e}^{H\delta t} \cdots \mathrm{e}^{H\delta t}$$

$$\mathrm{e}^{H\delta t} = \left(\prod_{\ell=1}^{N-1} \mathrm{e}^{h_{\ell,\ell+1}\delta t/2}\right)\left(\prod_{\ell=N-1}^{1} \mathrm{e}^{h_{\ell,\ell+1}\delta t/2}\right) + O\left(\delta t^3\right)$$

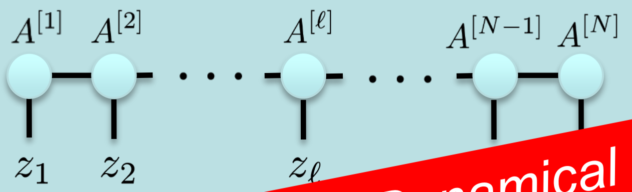
S_ℓ = two-site gate

Recall: Tensor Network



What I'm not going to talk about: Observables

Recall: Tensor Network

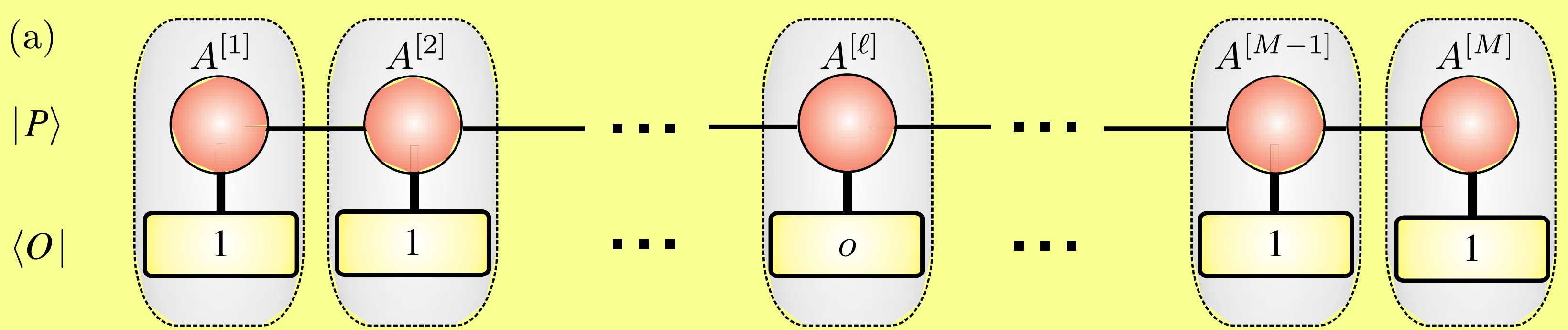


T. H. Johnson, S. R. Clark, and D. Jaksch, Dynamical simulation of stochastic systems using matrix product states, Phys. Rev. E **82**, 036702 (2010).

erages)

$$E[O,P(t)] = \langle O|P\rangle = \sum_{\mathbf{z}} O(\mathbf{z})P(\mathbf{z})$$

For a single-site observable



The competitor – Dynamical Monte Carlo

Expected value

$$\mathrm{E}\left[O,\mathcal{P}\right] = \int [\mathrm{D}\mathbf{s}]\, O[\mathbf{s}]\,\mathcal{P}[\mathbf{s}]$$

Sample paths $\mathbf{s}^1, \cdots, \mathbf{s}^M$

Average $\bar{O}_M = \frac{1}{M}\sum_{m=1}^{M} O[\mathbf{s}^m]$

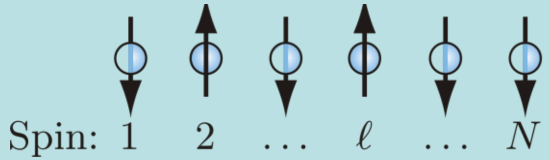
Typical error $\langle\overline{\Delta O_M}\rangle = \sqrt{\frac{\mathrm{Var}\left[O,\mathcal{P}\right]}{M}}$

Number of paths needed $M = \frac{\mathrm{Var}\left[O,\mathcal{P}\right]}{\mathcal{E}\mathrm{E}\left[O,\mathcal{P}\right]^2}$

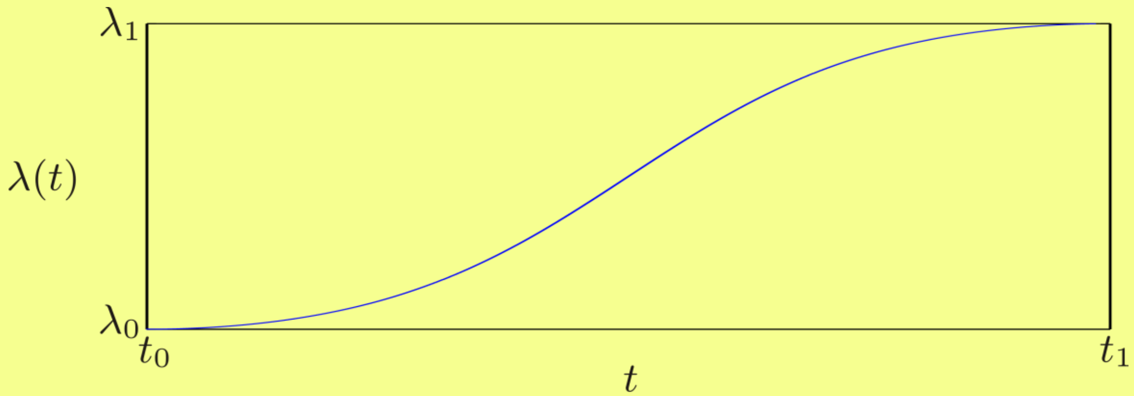
Large variance
=
Difficult

High variance example – Jarzynski process

Recall: *Ising chain*



Drive away from equilibrium



While it also thermalises

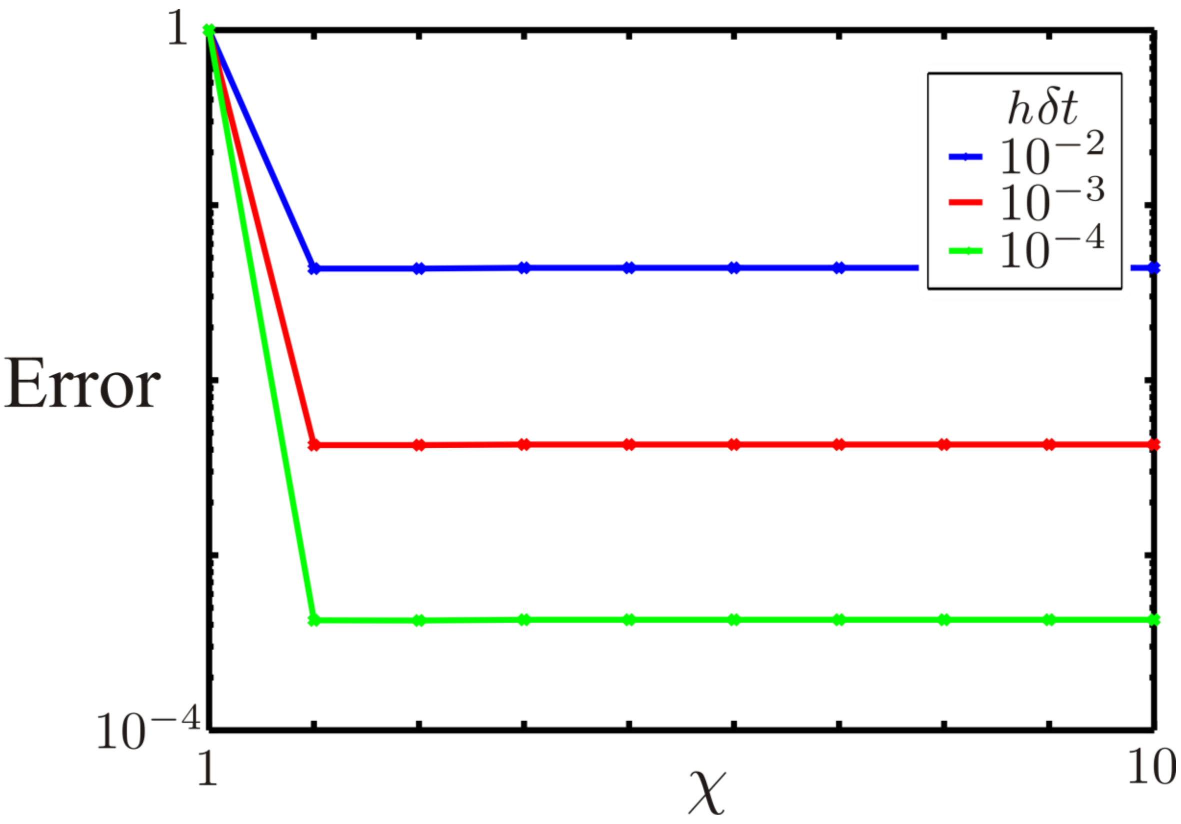
$$H_{\mathbf{z}\to\mathbf{z}'} = \frac{h}{1+\exp[-\beta(E(\mathbf{z})-E(\mathbf{z}'))]}$$

High variance observable

$$\exp[-\beta W[\mathbf{s}]] = \exp\left[-\beta\int_0^t \mathrm{d}t'\,\dot{\lambda}\frac{\mathrm{d}E(\mathbf{z})}{\mathrm{d}\lambda}\right]$$

Results (1)

Tensor networks fractional error



Parameters

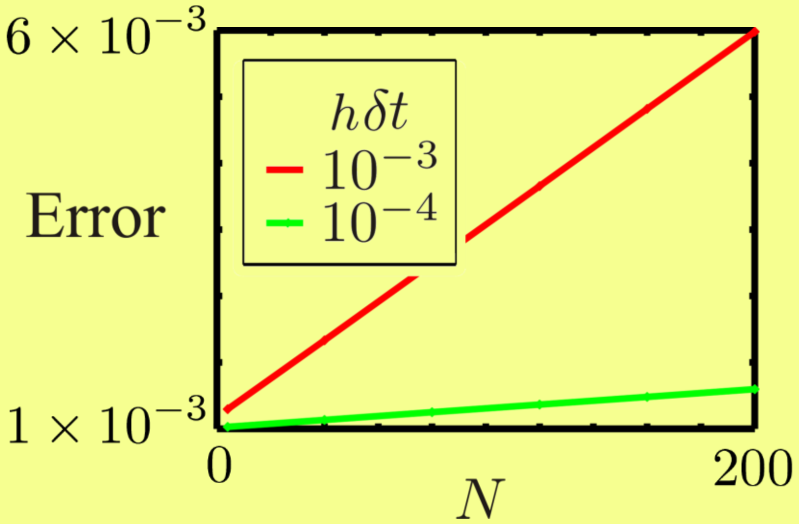
$$N = 200$$
$$h(t_1 - t_0) = 10$$
$$\beta J = 1$$
$$\beta \lambda_0 = 0$$
$$\beta \lambda_1 = 1$$

Highly compressible out of equilibrium

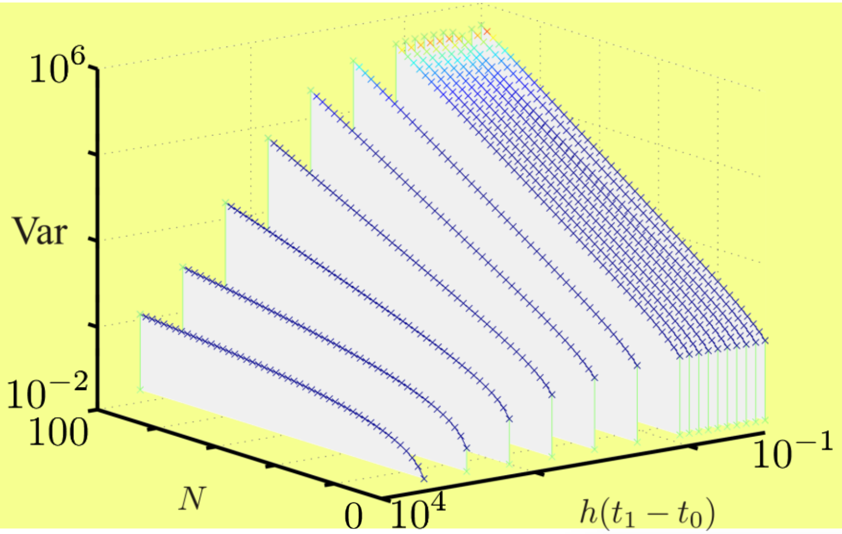
Results (2)

Recall: DMC and variance

$$M = \frac{\mathrm{Var}\left[O, \mathcal{P}\right]}{\mathcal{E}\mathrm{E}\left[O, \mathcal{P}\right]^2}$$



Exponentially increasing variance



Exponential samples needed to match tensor networks

Summary and future directions

Tensor networks compress data

In such a way as to allow its efficient updating

Tensor networks can simulate stochastic processes

Even when sampling is infeasible

Thanks for listening

Thomas Elliott

Stephen Clark

Dieter Jaksch

T. H. Johnson, S. R. Clark, and D. Jaksch, Phys. Rev. E **82**, 036702 (2010)

Preprint on arXiv soon

Tensor Network Library, tensornetworktheory.org

Questions?