INSTITUTE FOR SCIENTIFIC INTERCHANGE
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Simulating classical stochastic processes using quantum-inspired network-based data compression

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Overview

1. (a) Defining the problem

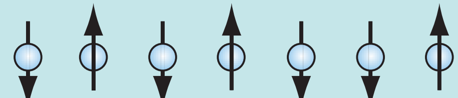
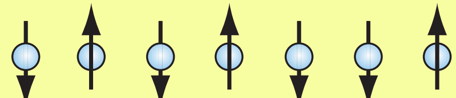
   (b) Connecting pure stochastic and pure quantum dynamics

2. Our method – Tensor networks

3. Main rival – Dynamical Monte Carlo

4. Comparison for an exponentially scaling variance

Representing the state of a system

Introducing the basics	Quantum (pure states)	Stochastic (probabilistic mixtures)
		
Local configs	$i_\ell = 0, 1, \ldots, d-1$	$i_\ell = 0, 1, \ldots, d-1$
Global configs	$\mathbf{i} = (i_1\, i_2\, \cdots\, i_N)$ $D = d^N$	$\mathbf{i} = (i_1\, i_2\, \cdots\, i_N)$ $D = d^N$
State vector	$\lvert\psi\rangle = \sum_{\mathbf{i}} \psi_{\mathbf{i}} \lvert\mathbf{i}\rangle$	$\lvert P\rangle = \sum_{\mathbf{i}} P_{\mathbf{i}} \lvert\mathbf{i}\rangle$
Probabilities	$\lvert\psi_{\mathbf{i}}\rvert^2$	$P_{\mathbf{i}}$

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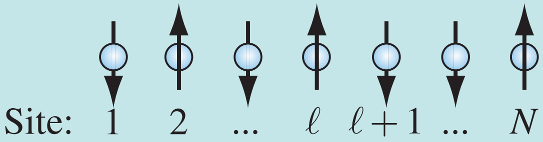
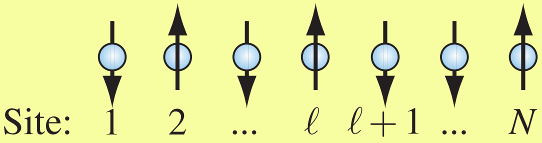
Evolving the state of a system

Introducing the basics

	Unitary evolution	Markovian stochastic process
		
Master equation		$$\frac{\partial P_{\mathbf{i}}(t)}{\partial t} = \sum_{\mathbf{i}' \neq \mathbf{i}} \left(P_{\mathbf{i}'}(t) R_{\mathbf{i}' \to \mathbf{i}} - P_{\mathbf{i}}(t) R_{\mathbf{i} \to \mathbf{i}'} \right)$$
Rearrangement		$$\langle \mathbf{i} \vert H \vert \mathbf{i}' \rangle = R_{\mathbf{i}' \to \mathbf{i}} \quad \text{for} \quad \mathbf{i} \neq \mathbf{i}',$$ $$\langle \mathbf{i} \vert H \vert \mathbf{i} \rangle = -\sum_{\mathbf{i}' \neq \mathbf{i}} R_{\mathbf{i} \to \mathbf{i}'}.$$
Evolution equation	$$\mathrm{i}\hbar \frac{\partial}{\partial t} \vert \psi(t) \rangle = H \vert \psi(t) \rangle$$	$$\frac{\partial}{\partial t} \vert P(t) \rangle = H \vert P(t) \rangle$$
Evolution operator	$$\vert \psi(t) \rangle = \mathrm{e}^{-\mathrm{i}Ht/\hbar} \vert \psi(0) \rangle$$	$$\vert P(t) \rangle = \mathrm{e}^{Ht} \vert P(0) \rangle$$

Solving the representation problem

Recall: Dimension explosion

$$D = d^N$$

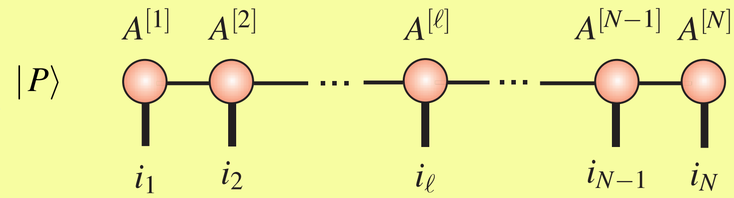
Mean Field $P_{\mathbf{i}} = P_{i_1}^{[1]} P_{i_2}^{[2]} \cdots P_{i_N}^{[N]}$

Matrix Product State

$$P_{\mathbf{i}} = A_{i_1}^{[1]} \times A_{i_2}^{[2]} \times \cdots \times A_{i_N}^{[N]}$$

$(1 \times \chi)$ $(\chi \times \chi)$ $(\chi \times 1)$

MPS as a tensor network



Solving the evolution problem I

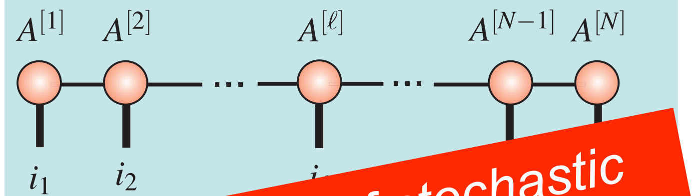
Recall: *Evolution operator*

$$|P(t)\rangle = \mathrm{e}^{Ht}|P(0)\rangle$$

Assume: *Nearest-neighbour*

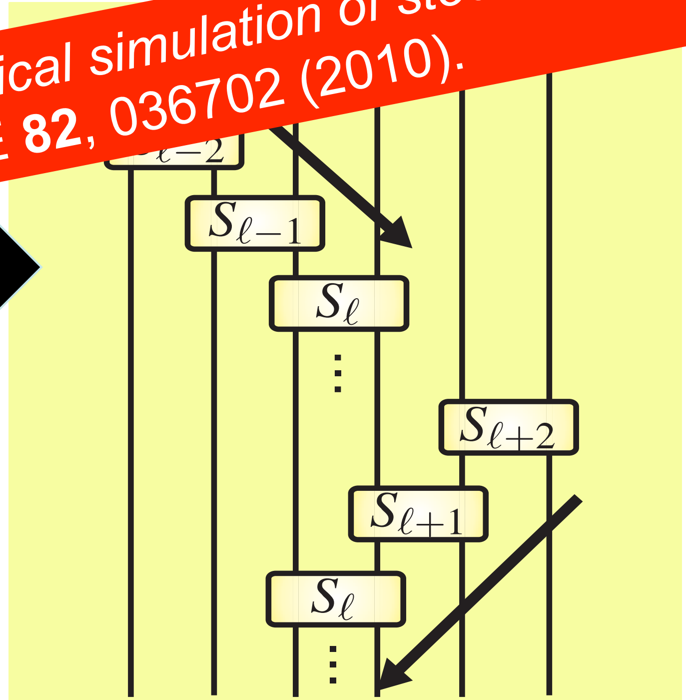
$$H = \sum_{\ell=1}^{N-1} h_{\ell,\ell+1},$$

Recall: *Tensor Network*



T. H. Johnson, S. R. Clark, and D. Jaksch, Dynamical simulation of stochastic systems using matrix product states, Phys. Rev. E **82**, 036702 (2010).

e^{Ht}



$$\mathrm{e}^{H\delta t} = \left(\prod_{\ell=1}^{N-1} \mathrm{e}^{h_{\ell,\ell+1}\delta t/2}\right)\left(\prod_{\ell=N-1}^{1} \mathrm{e}^{h_{\ell,\ell+1}\delta t/2}\right) + O\left(\delta t^3\right)$$

S_ℓ = two-site gate

The competitor – Dynamical Monte Carlo

Define: *general expected values*

$$\mathrm{E}\left[O,\mathcal{P}\right] = \int [\mathrm{D}\mathbf{s}] O[\mathbf{s}] \mathcal{P}[\mathbf{s}]$$

Sample paths $\mathbf{s}^1, \cdots, \mathbf{s}^M$

Average $\bar{O}_M = \frac{1}{M}\sum_{m=1}^{M} O[\mathbf{s}^m]$

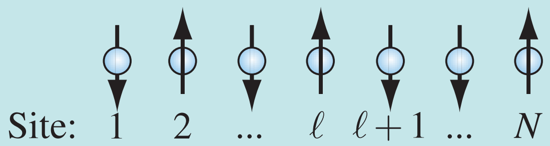
Typical error $\langle\overline{\Delta O_M}\rangle = \sqrt{\frac{\cdots}{M}}$

Number of paths needed $M = \frac{\mathrm{Var}\left[O,\mathcal{P}\right]}{\mathcal{E}\mathrm{E}\left[O,\mathcal{P}\right]^2}$

Large variance
=
Difficult

High variance example – Jarzynski process

Recall: *Ising model*



Definition, energy

$$E_{\mathbf{i}} = -J \sum_{\langle \ell \ell' \rangle} (2i_\ell - 1)(2i_{\ell'} - 1) - B(t) \sum_{\ell} (2i_\ell - 1)$$

Thermalising process

$$R_{\mathbf{i} \to \mathbf{i}'} = \frac{r}{1 + \mathrm{e}^{-\beta(E_{\mathbf{i}} - E_{\mathbf{i}'})}}$$

High variance expected value

$$\mathrm{e}^{-\beta W[\mathbf{s}]} = \exp\left(-\beta \int_0^t \mathrm{d}t \, \dot{w}_{\mathbf{s}(t)}\right)$$

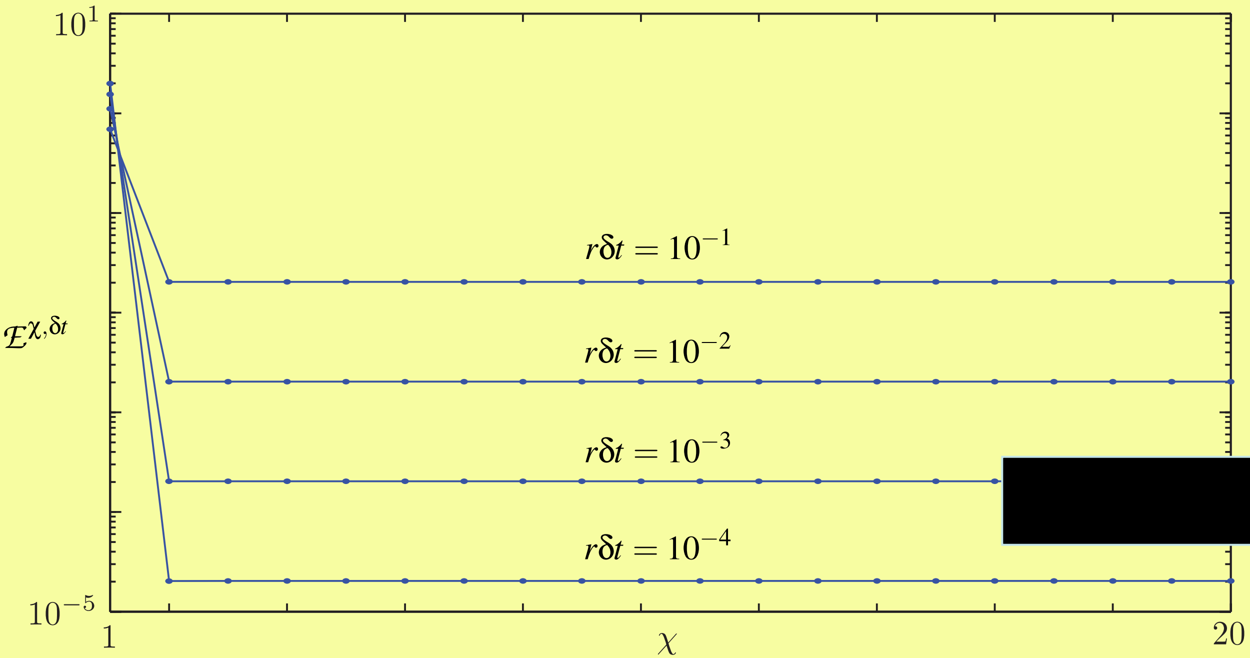
$$\dot{w}_{\mathbf{i}} = -\dot{B} \sum_{\mathbf{i}} (2i_\ell - 1)$$

Results I

Example: *linearly increasing field*

$$\beta B(t) = t/\tau$$

Tensor networks fractional error



Parameters

$$N = 40$$

$$r\tau = 10$$

$$\beta J = 1$$

Highly compressible

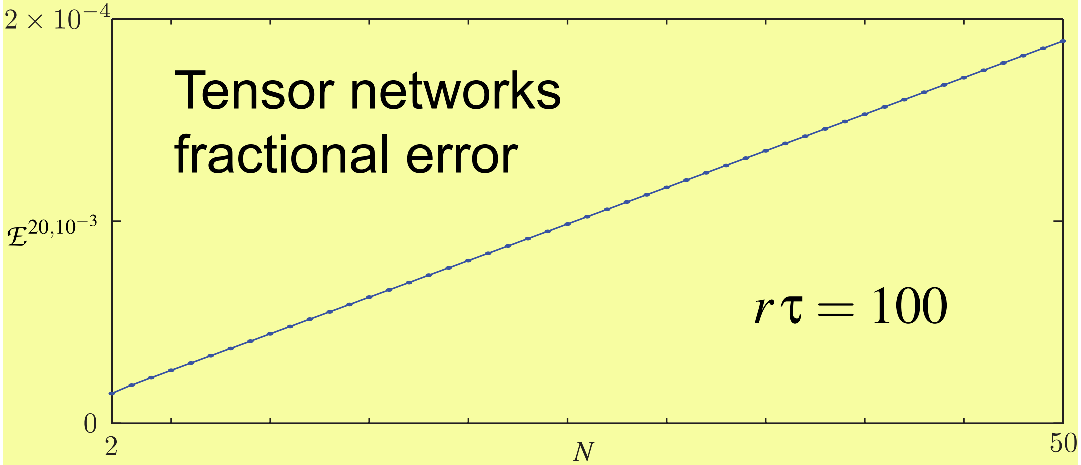
Results II

Recall: DMC and variance

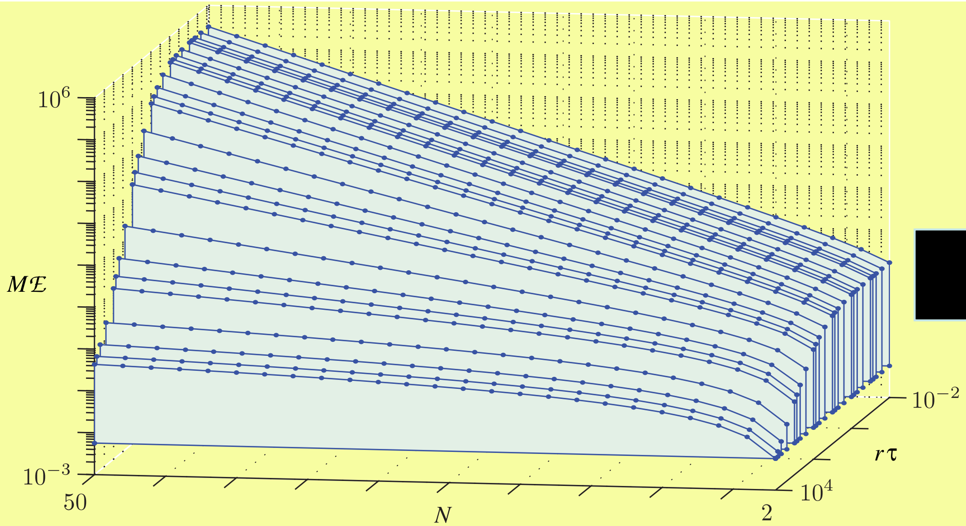
$$M = \frac{\mathrm{Var}[O,\mathcal{P}]}{\mathcal{E}\mathrm{E}[O,\mathcal{P}]^2}$$

Recall: linear ramp

$$\beta B(t) = t/\tau$$



Exponentially increasing variance



Tensor neworks give an exponential speed-up

Summary and future directions

1. Tensor networks simulate stochastic processes

2. Is this the way to estimate high variance observables?

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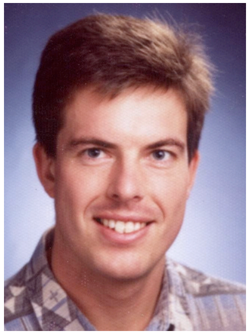
1. Higher dimensions?

2. What else can we do with tensor networks – PDEs?

Thanks for listening

Thomas Elliott

Stephen Clark



Dieter Jaksch

T. H. Johnson, S. R. Clark, and D. Jaksch, Phys. Rev. E **82**, 036702 (2010)

Tensor Network Library, tensornetworktheory.org

Questions?