CLARENDON LABORATORY
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Tensor networks and stochastic systems: Using quantum methods to model traffic

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Overview

Stochastic systems – *why?*

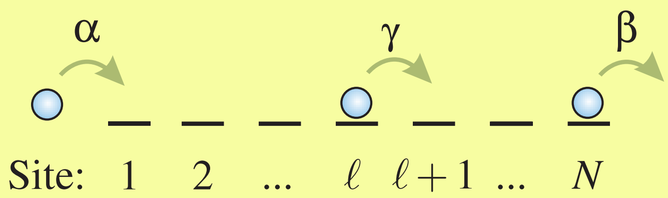
The connection between stochastic and quantum – *what?*

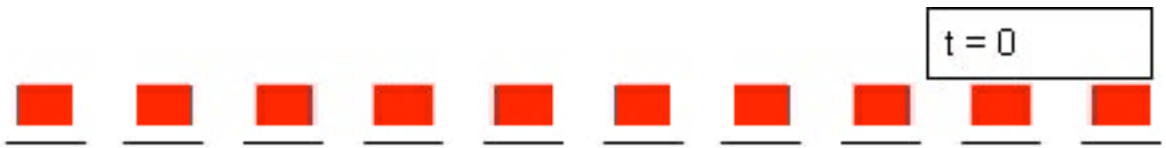
Tensor network methods – *how?*

Traffic jams – *example*

A paradigm

Totally Asymmetric Exclusion Process (TASEP)



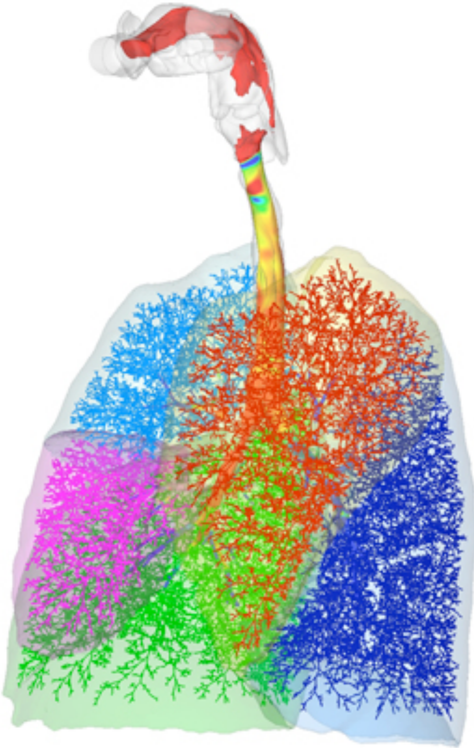


On average

$$\frac{\partial P_{\mathbf{i}}(t)}{\partial t} = \sum_{\mathbf{i}' \neq \mathbf{i}} \left(P_{\mathbf{i}'}(t) R_{\mathbf{i}' \to \mathbf{i}} - P_{\mathbf{i}}(t) R_{\mathbf{i} \to \mathbf{i}'} \right)$$

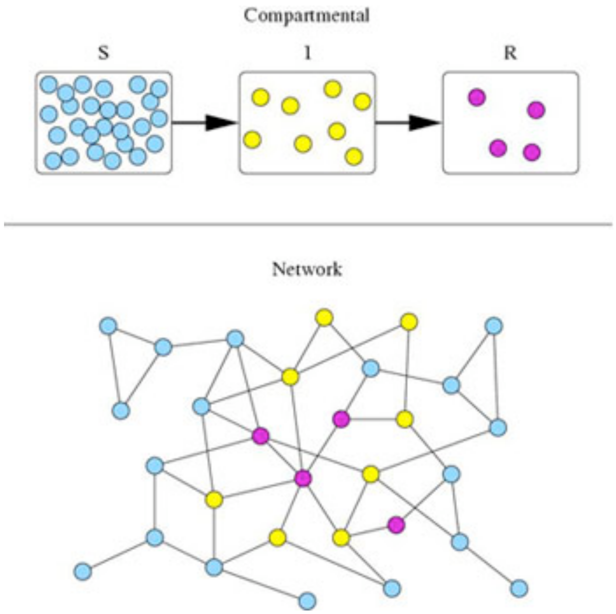
Complex systems

1D trees



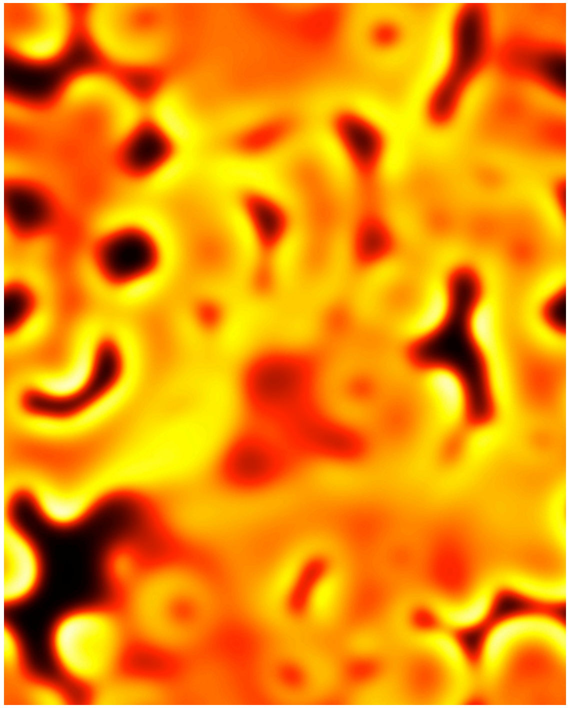
Queues
Production line

Random graphs



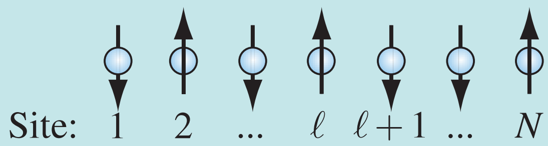
Epidemics
Social networks

Two dimensions



Chemical reactions
2D spin systems

Structure

Recall: Standard quantum formalism	Quantum (pure states)	Stochastic
		
Local configs	$i_\ell = 1, 2, \ldots, d$	$i_\ell = 1, 2, \ldots, d$
Global configs	$\mathbf{i} = (i_1\, i_2\, \cdots\, i_N)$ $D = d^N$	$\mathbf{i} = (i_1\, i_2\, \cdots\, i_N)$ $D = d^N$
State vector	$\lvert\psi\rangle = \sum_{\mathbf{i}} \psi_{\mathbf{i}} \lvert\mathbf{i}\rangle$	$\lvert P\rangle = \sum_{\mathbf{i}} P_{\mathbf{i}} \lvert\mathbf{i}\rangle$
Probabilities	$\lvert\psi_{\mathbf{i}}\rvert^2$	$P_{\mathbf{i}}$

Evolution

Recall: Master equation

$$\frac{\partial P_{\mathbf{i}}(t)}{\partial t} = \sum_{\mathbf{i}' \neq \mathbf{i}} \left(P_{\mathbf{i}'}(t) R_{\mathbf{i}' \to \mathbf{i}} - P_{\mathbf{i}}(t) R_{\mathbf{i} \to \mathbf{i}'} \right)$$

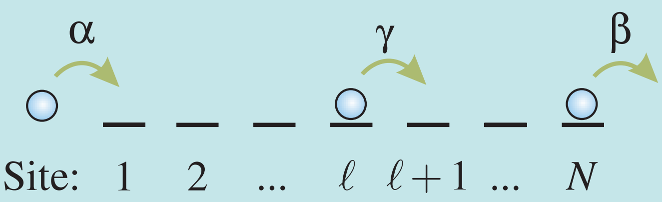
Equivalent to

$$\frac{\partial}{\partial t} | P(t) \rangle = H | P(t) \rangle$$

$$\langle \mathbf{i} | H | \mathbf{i}' \rangle = R_{\mathbf{i}' \to \mathbf{i}} \quad \text{for} \quad \mathbf{i} \neq \mathbf{i}',$$

$$\langle \mathbf{i} | H | \mathbf{i} \rangle = - \sum_{\mathbf{i}' \neq \mathbf{i}} R_{\mathbf{i} \to \mathbf{i}'} .$$

Recall: TASEP



TASEP has single site and nearest-neighbour terms only.

Tensor Network

Recall: Dimension explosion

$$D = d^N$$

Recall: Evolution operator

$$|P(t)\rangle = \mathrm{e}^{Ht}|P(0)\rangle$$

Recall: Nearest-neighbour

$$H = \sum_{\ell=1}^{N-1} h_{\ell,\ell+1},$$

Mean Field $P_{\mathbf{i}} = P_{i_1}^{[1]} P_{i_2}^{[2]} \cdots P_{i_N}^{[N]}$

Matrix Product State

$$P_{\mathbf{i}} = A_{i_1}^{[1]} \times A_{i_2}^{[2]} \times \cdots \times A_{i_N}^{[N]}$$

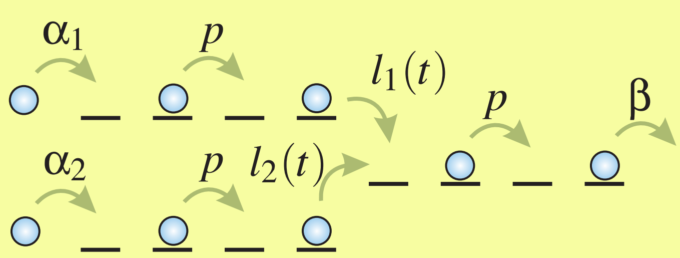
$$\mathrm{e}^{Ht} = \mathrm{e}^{H\delta t}\mathrm{e}^{H\delta t}\cdots\mathrm{e}^{H\delta t}$$

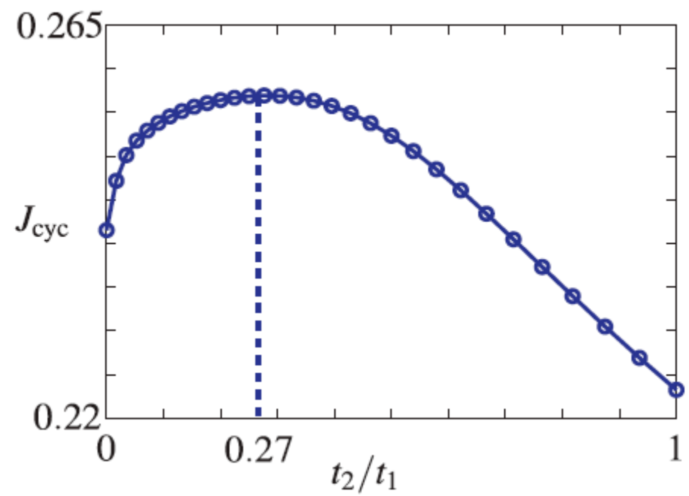
$$\mathrm{e}^{H\delta t} = \left(\prod_{\ell=1}^{N-1} \mathrm{e}^{h_{\ell,\ell+1}\delta t/2}\right)\left(\prod_{\ell=N-1}^{1} \mathrm{e}^{h_{\ell,\ell+1}\delta t/2}\right)$$

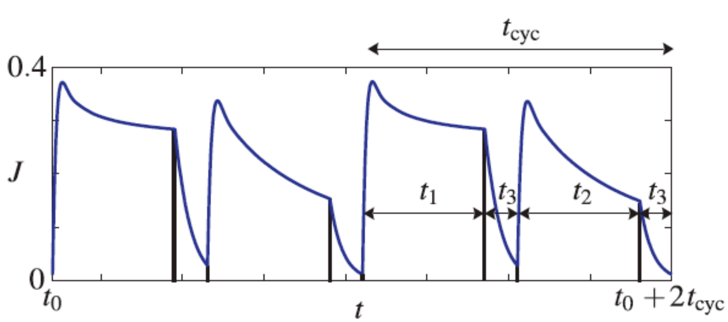
T. H. Johnson, S. R. Clark, and D. Jaksch, *Dynamical simulation of stochastic systems using matrix product states,* Phys. Rev. E **82**, 036702 (2010).

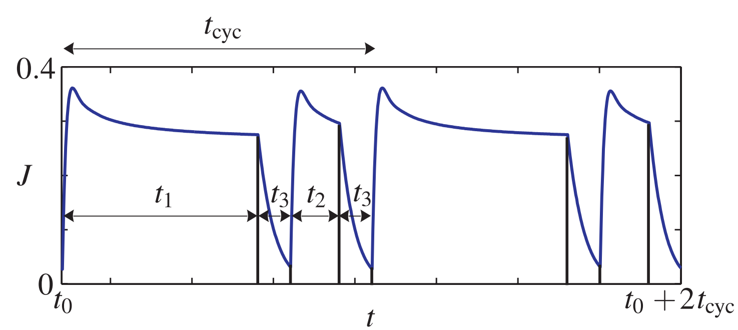
Simulating traffic

Time evolution example:
Traffic light model



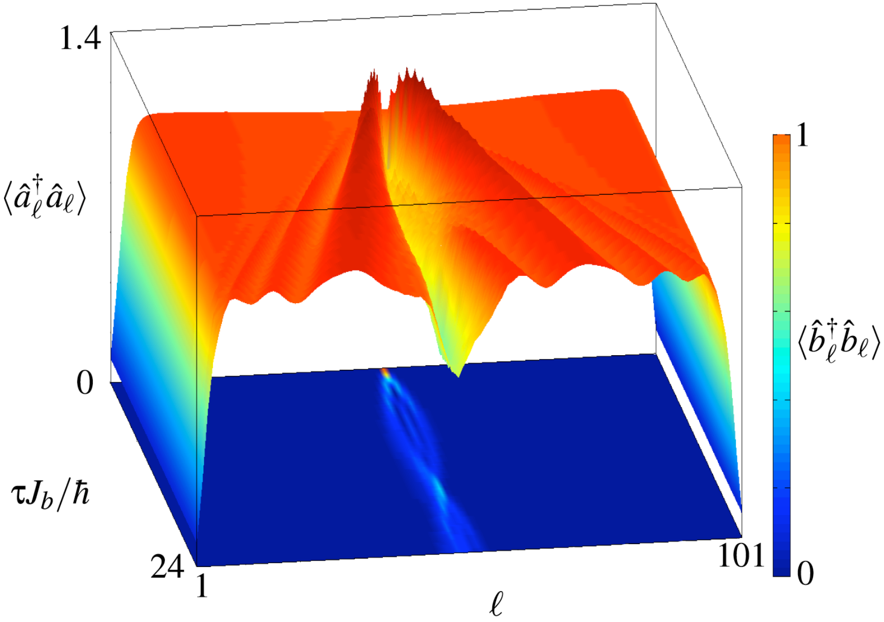






Summary and further work

Quantum



M. Bruderer, T. H. Johnson, S. R. Clark, D. Jaksch, A. Posazhennikova, and W. Belzig, Phys. Rev. A (to appear).

Image compression

Finite difference methods
Partial differential equations

Thanks for listening

Any questions?